Second-Order Linear Equations

A second-order linear differential equation on I can be written as

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = G(x), \quad x \in I,$$

where P, Q and G are arbitrary functions of the independent variable $x \in I$. Particularly important are the constant-coefficient equations, where P and Q (but not necessarily G) are constants, and the homogeneous equations, where $G(x) = 0$ for all $x \in I$.

Thus the form of a second-order linear homogeneous differential equation is

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

If $G(x) \neq 0$ for some $x \in I$, it is called a nonhomogeneous differential equation.

Definition A general solution to a second-order linear differential equation is a solution containing two arbitrary constants of integration. A particular solution is derived from the general solution by setting the constants of integration to values that satisfy the initial value conditions of the problem.

Definition Two functions y_1 and y_2 are said to be linearly independent in I if neither y_1 nor y_2 is a constant multiple of the other throughout I.

Remark Two differentiable functions y_1 and y_2 are linearly independent in $I = (a, b)$ if and only if

$$\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \det \begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix} = y_1(x)y_2'(x) - y_2(x)y_1'(x) = 0 \quad \text{for all } x \in I.$$

Proof If y_1 and y_2 are linearly dependent, then there exists a constant $c \in \mathbb{R}$ such that $y_2(x) = cy_1(x)$ for each $x \in I$ which implies that

$$\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} y_1(x) & cy_1(x) \\ y_1'(x) & cy_1'(x) \end{vmatrix} = cy_1(x)y_1'(x) - cy_1(x)y_1'(x) = 0 \quad \text{for all } x \in I.$$

Conversely, if $y_1(x)y_2'(x) - y_2(x)y_1'(x) = 0$ for all $x \in I$, then

$$\frac{y_1'(x)}{y_1(x)} = \frac{y_2'(x)}{y_2(x)} \quad \text{whenever } y_1(x), y_2(x) \neq 0 \implies y_2(x) = cy_1(x) \quad \text{for } x \in I,$$

which implies that y_1 and y_2 are linearly dependent in I.

Example The functions $f(x) = x^2$ and $g(x) = 2x^2$ are linearly dependent, but $f(x) = e^x$ and $g(x) = xe^x$ are linearly independent.

Principle of Superposition If $y_1(x)$ and $y_2(x)$ are solutions of the linear homogeneous differential equation

$$(*) \quad \frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \quad x \in I,$$

and if c_1 and c_2 are constants, then the linear combination

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

is also a solution of the linear homogeneous differential equation $(*)$.

Theorem 1 If $y_1(x)$ and $y_2(x)$ are linearly independent solutions of the linear homogeneous differential equation

$$(*) \quad \frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \quad x \in I,$$

then the general solution $(*)$ is given by the linear combination

$$y(x) = c_1y_1(x) + c_2y_2(x), \quad \text{where } c_1, c_2 \text{ are arbitrary constants.}$$

Theorem 2 If $y_1(x)$ and $y_2(x)$ are linearly independent solutions of

$$(*) \quad \frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \quad x \in I,$$

and if $y_p(x)$ is a particular solution of

$$(\dagger) \quad \frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = G(x), \quad x \in I,$$

then the general solution of $(\dagger)$ is given by

$$y(x) = y_p(x) + c_1y_1(x) + c_2y_2(x) = \text{a linear combination of } y_1 \text{ and } y_2$$

where c_1 and c_2 are arbitrary constants.

Remark The space of solutions for a second order linear differential equation $(\dagger)$ can be viewed as a space parametrized by $c_1, c_2 \in \mathbb{R}$ and is a space of dimension 2. Thus if we know two particular linearly independent solutions, then we know every solution.

In general, it's not easy to discover particular solutions to a second-order linear equation. But it is always possible to do so for a second order linear homogeneous equation with constant coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

where $a \neq 0$, b and c are constants.

Since $y = e^{rx}$ (where r is a constant) has the property that its derivative is a constant multiple of itself: $y' = re^{rx}$. Furthermore, $y'' = r^2e^{rx}$. If we substitute these expressions into the above second-order constant coefficients differential equation we see that $y = e^{rx}$ is a solution if

$$(ar^2 + br + c)e^{rx} = 0 \iff ar^2 + br + c = 0 \quad \text{since } e^{rx} \neq 0 \text{ for all } x,$$

where the algebraic equation $ar^2 + br + c = 0$ is called the auxiliary equation (or characteristic equation) of the differential equation $ay'' + by' + cy = 0$.

Definition The polar form of a complex number expresses a number in terms of an angle θ and its distance from the origin r. Given a complex number in rectangular form expressed as $z = x + iy$, since

$$x = r\cos\theta \quad y = r\sin\theta \quad r = \sqrt{x^2 + y^2},$$

we have

$$z = x + iy = r(\cos\theta + i\sin\theta),$$

where r is the modulus and θ is the argument. We often use the abbreviation $r\text{cis}\,\theta$ to represent $r(\cos\theta + i\sin\theta)$.

Since

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2),$$

we define

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \text{for } \theta\in\mathbb{R} \implies |e^{i\theta}| = 1 \text{ for all } \theta\in\mathbb{R} \text{ and } e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}.$$

If $z = x+iy$ for $x,\, y\in\mathbb{R}$, then

$$\begin{aligned} z &= x+iy = r(\cos\theta + i\sin\theta) = re^{i\theta}, \quad \text{where } r = |z| = \sqrt{x^2+y^2},\ x = r\cos\theta,\ y = r\sin\theta \\ e^z &= e^{x+iy} = e^x(\cos y + i\sin y) \implies |e^z| = e^x. \end{aligned}$$

Examples Find the polar form of $(a)\ z = 4i$, $(b)\ z = -4+4i$, $(c)\ z = \sqrt{3}+i$.

Theorem The general solution of the differential equation $ay'' + by' + cy = 0$ is

(1) $y = c_1e^{r_1x} + c_2e^{r_2x}$ when $b^2-4ac > 0$ and $r_1 \neq r_2 \in \mathbb{R}$ are two distinct real roots of $ar^2+br+c = 0$ given by

$$r_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}, \quad r_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}.$$

(2) $y = c_1e^{r_1x} + c_2e^{r_2x} = e^{\alpha x}[c_3\cos\beta x + ic_4\sin\beta x]$ when $b^2-4ac<0$ and $r_1 = \alpha+i\beta \neq r_2 = \alpha - i\beta \in \mathbb{C}$ are two distinct complex roots of $ar^2+br+c=0$ given by

$$r_1 = \alpha+i\beta = \frac{-b+i\sqrt{4ac-b^2}}{2a},\ r_2 = \alpha-i\beta = \frac{-b-i\sqrt{4ac-b^2}}{2a} = \bar{r}_1 = \text{complex conjugate of } r_1.$$

Note that

$$\begin{aligned} y &= c_1e^{r_1x} + c_2e^{r_2x} = c_1e^{(\alpha+i\beta)x} + c_2e^{(\alpha-i\beta)x} \\ &= c_1e^{\alpha x}(\cos\beta x + i\sin\beta x) + c_2e^{\alpha x}(\cos\beta x - i\sin\beta x) \\ &= e^{\alpha x}[(c_1+c_2)\cos\beta x + i(c_1-c_2)\sin\beta x] \\ &= e^{\alpha x}[c_3\cos\beta x + ic_4\sin\beta x], \quad \text{where } c_3 = c_1+c_2,\ c_4 = c_1-c_2. \end{aligned}$$

(3) $y = c_1e^{rx} + c_2xe^{rx}$ when $b^2-4ac = 0$ and r is the only real root of $ar^2+br+c=0$.

Remarks

1. Let $r_1 = \dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $r_2 = \dfrac{-b-\sqrt{b^2-4ac}}{2a}$. Since

$$\begin{aligned} 0 &= ay''+by'+cy = a\left[\frac{d^2y}{dx^2} - (r_1+r_2)\frac{dy}{dx} + r_1r_2y\right] \\ &= a\left(\frac{d}{dx} - r_2\right)\left(\frac{dy}{dx} - r_1y\right) \overset{\text{or}}{=} a\left(\frac{d}{dx}-r_1\right)\left(\frac{dy}{dx} - r_2y\right), \end{aligned}$$

$y_1 = e^{r_1x}$ and $y_2 = e^{r_2x}$ are solutions of the differential equation $ay''+by'+cy=0$.

2. If $b^2-4ac = 0$, since

$$0 = ar^2+br+c = a(r^2 + \frac{b}{a}r + \frac{c}{a}) = a(r+\frac{b}{2a})^2 \implies r = -\frac{b}{2a},$$

$y_1 = e^{rx}$ is a solution of $ay'' + by' + cy = 0$.
To find a 2^{nd} linearly independent solution of $ay'' + by' + cy = 0$, we set $y_2 = vy_1$ and substitute it into the equation to get

$$\begin{aligned}
0 &= ay_2'' + by_2' + cy_2 = a(vy_1)'' + b(vy_1)' + c(vy_1) \\
&= a(vy_1'' + 2v'y_1' + v''y_1) + b(vy_1' + v'y_1) + c(vy_1) \\
&= v(ay_1'' + by_1' + cy_1) + (2av'y_1' + bv'y_1) + v''y_1 \\
&= (2av'ry_1 + bv'y_1) + v''y_1 = (2ar + b)v'y_1 + v''y_1 \\
&= v''y_1 \\
\Longrightarrow\ & v'' = 0 \Longrightarrow v = c_1x + c_2 \text{ and } c_1 \neq 0.
\end{aligned}$$

Hence $y_2 = xy_1$ is a 2^{nd} linearly independent solution.

Examples Solve the differential equation

(1) $y'' + y' - 6y = 0$.

(2) $3y'' + y' - y = 0$.

(3) $4y'' + 12y' + 9y = 0$.

(4) $y'' - 6y' + 13y = 0$.

An initial-value problem for the second-order differential equation $P(x)\dfrac{d^2y}{dx^2} + Q(x)\dfrac{dy}{dx} + R(x)y = G(x)$, $x \in I$, consists of finding a solution y of the differential equation that also satisfies initial conditions of the form

$$y(x_0) = y_0, \quad y'(x_0) = y_1 \quad \text{for some } x_0 \in I,$$

where y_0 and y_1 are given constants. If P, Q, R and G are continuous on I and $P(x) \neq 0$ for $x \in I$, then a theorem found in more advanced books guarantees the existence and uniqueness of a solution to this initial-value problem.

Example Solve the initial-value problem

$$y'' + y' - 6y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

A boundary-value problem for the second-order differential equation $P(x)\dfrac{d^2y}{dx^2} + Q(x)\dfrac{dy}{dx} + R(x)y = G(x)$, $x \in I$, consists of finding a solution y of the differential equation that also satisfies boundary conditions of the form

$$y(x_0) = y_0, \quad y(x_1) = y_1 \quad \text{where } x_0,\ x_1 \text{ are boundary points (or end points) of } I.$$

In contrast with the situation for initial-value problems, a boundary-value problem does not always have a solution.

Example Solve the boundary-value problem $y'' + 2y' + y = e^{-x}$, $y(0) = 1$, $y(1) = 3$.

Solution : Since the characteristic equation $0 = r^2 + 2r + 1 = (r+1)^2$ has a double root $r = -1$, $y_1(x) = e^{-x}$ and $y_2(x) = xe^{-x}$ are linearly independent homogeneous solutions of $y'' + 2y' + y = 0$. To find a particular solution y_p of $y'' + 2y' + y = e^{-x}$, we set $y_p(x) = Ax^2e^{-x}$ and use the method of undetermined coefficients to get $A = \dfrac{1}{2}$ and the general solution $y(x) = \dfrac{1}{2}x^2e^{-x} + C_1e^{-x} + C_2xe^{-x}$, where C_1, C_2 are arbitrary constants. Using the boundary conditions $y(0) = 1$ and $y(1) = 3$, we get $C_1 = 1$, $C_2 = 3e - \dfrac{3}{2}$, and $y(x) = \dfrac{1}{2}x^2e^{-x} + e^{-x} + (3e - \dfrac{3}{2})xe^{-x}$.

Theorem Consider the second-order nonhomogeneous linear differential equations with constant coefficients

$$(*) \qquad ay'' + by' + cy = G(x), \quad x \in I$$

where a, b and c are constants and $G(x)$ is continuous for $x \in I$.

1. If $y_{p_1}(x)$ and $y_{p_2}(x)$ are two (particular) solutions of $(*)$, then $y_{p_1}(x) - y_{p_2}(x)$ is a (homogeneous) solution of the (homogeneous) equation

$$ay'' + by' + cy = 0, \quad x \in I.$$

2. The general solution of $(*)$ can be written as

$$y(x) = y_p(x) + y_h(x),$$

where $y_p(x)$ is a particular solution of $(*)$ and $y_h(x)$ is the general (homogeneous) solution of the homogeneous equation

$$ay'' + by' + cy = 0, \quad x \in I.$$

The Method of Undetermined Coefficients is used to find a particular solution of the second-order nonhomogeneous linear differential equations with constant coefficients

$$(*) \qquad ay'' + by' + cy = G(x), \quad x \in I.$$

1. If $G(x) = e^{kx}P(x)$, where $P(x)$ is a polynomial of degree n, then try

$$y_p(x) = e^{kx}Q(x),$$

where $Q(x)$ is an n^{th}-degree polynomial (whose coefficients are determined by substituting in the differential equation).

2. If $G(x) = e^{kx}P(x)\cos mx$ or $G(x) = e^{kx}P(x)\sin mx$, where $P(x)$ is an n^{th}-degree polynomial, then try

$$y_p(x) = e^{kx}Q(x)\cos mx + e^{kx}R(x)\sin mx,$$

where $Q(x)$, $R(x)$ are n^{th}-degree polynomials.

Modification: If any term of y_p is a solution of the homogeneous differential equation, multiply y_p by x (or by x^2 if necessary).

Example Solve the initial-value problem $y'' + y' - 2y = x^2 + \sin x + e^x$, $y(0) = 1$, $y'(0) = 2$.

Solution : Since the characteristic equation

$$0 = r^2 + r - 2 = (r+2)(r-1)$$

has roots $r = -2$ or 1, $y_1(x) = e^{-2x}$ and $y_2(x) = e^x$ are linearly independent homogeneous solutions of

$$y'' + y' - 2y = 0.$$

To find a particular solution y_p of $y'' + y' - 2y = x^2 + \sin x + e^x$, we set

$$y_p(x) = A_2x^2 + A_1x + A_0 + B\cos x + C\sin x + Dxe^x$$

and use the method of undetermined coefficients to get $A_2 = A_1 = -\frac{1}{2}$, $A_0 = -\frac{3}{4}$, $B = -\frac{1}{10}$, $C = -\frac{3}{10}$, $D = \frac{1}{3}$, and the general solution

$$y(x) = -\frac{x^2}{2} - \frac{x}{2} - \frac{3}{4} - \frac{1}{10}\cos x - \frac{3}{10}\sin x + \frac{1}{3}xe^x + C_1e^{-2x} + C_2e^x,$$

where C_1, C_2 are arbitrary constants. Using the initial conditions $y(0) = 1$ and $y'(0) = 2$, we obtain $C_1 = -\frac{37}{180}$, $C_2 = \frac{37}{18}$, and the solution $y(x) = y_p(x) - \frac{37}{180}e^{-2x} + \frac{37}{18}e^x$.

The Method of Variation of Parameters is used to find a particular solution of the nonhomogeneous equation $ay'' + by' + cy = G(x)$, $x \in I$, of the form

$$(\dagger) \qquad y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) \quad x \in I,$$

where y_1 and y_2 are linearly independent (homogeneous) solutions of the (homogeneous) equation

$$ay'' + by' + cy = 0.$$

This method is called variation of parameters because we have varied the parameters c_1 and c_2 to make them functions.

Differentiating Equation ($\dagger$), we get

$$(\dagger') \qquad y_p' = (u_1'y_1 + u_2'y_2) + (u_1y_1' + u_2y_2'), \quad x \in I.$$

Since u_1 and u_2 are arbitrary functions, we can impose two conditions on them.

One condition is that y_p is a solution of the differential equation; we can choose the other condition so as to simplify our calculations. In view of the expression in Equation ($\dagger'$), let's impose the condition that

$$(\dagger\dagger) \qquad u_1'y_1 + u_2'y_2 = 0, \quad x \in I.$$

Substituting ($\dagger\dagger$) into ($\dagger'$), we get $y_p' = u_1'y_1 + u_2'y_2 + u_1y_1' + u_2y_2' = u_1y_1' + u_2y_2'$ for $x \in I$ and that

$$(\dagger'') \qquad y_p'' = u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2'', \quad x \in I.$$

Substituting ($\dagger$), ($\dagger'$) and ($\dagger''$) in the differential equation $ay'' + by' + cy = G(x)$, $x \in I$, we get

$$\begin{aligned} & a(u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2'') + b(u_1y_1' + u_2y_2') + c(u_1y_1 + u_2y_2) = G \\ \implies\ & u_1(ay_1'' + by_1' + cy_1) + u_2(ay_2'' + by_2' + cy_2) + a(u_1'y_1' + u_2'y_2') = G \\ \implies\ & (\dagger\dagger') \qquad a(u_1'y_1' + u_2'y_2') = G \quad \text{since } y_1,\ y_2 \text{ are homogeneous solutions.} \end{aligned}$$

Thus u_1', u_2' are solutions of the system ($\dagger\dagger$), ($\dagger\dagger'$)

$$\begin{cases} u_1'y_1 + u_2'y_2 = 0 \\ u_1'y_1' + u_2'y_2' = \dfrac{G}{a} \end{cases} \implies \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{G}{a} \end{pmatrix} \implies \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \frac{1}{y_1y_2' - y_2y_1'} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix} \begin{pmatrix} 0 \\ \dfrac{G}{a} \end{pmatrix}$$

Example Solve the differential equation $y'' + y = \tan x$, $0 < x < \frac{\pi}{2}$.

Solution : Since the characteristic equation $r^2 + 1 = 0$ has roots $r = \pm i$, $y_1(x) = \cos x$ and $y_2(x) = \sin x$ are linearly independent homogeneous solutions.

Let $y_p = u_1 y_1 + u_2 y_2$ be a particular solution with u_1', u_2' satisfying

$$\begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{\sin x}{\cos x} \end{pmatrix}$$

$$\implies \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} \dfrac{-\sin^2 x}{\cos x} \\ \sin x \end{pmatrix} = \begin{pmatrix} -\sec x + \cos x \\ \sin x \end{pmatrix}$$

$$\implies \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -\ln|\sec x + \tan x| + \sin x \\ -\cos x \end{pmatrix}$$

Then the general solution is

$$\begin{aligned} y &= (-\ln|\sec x + \tan x| + \sin x)\cos x - \cos x \sin x + C_1 \cos x + C_2 \sin x \\ &= -\cos x \ln|\sec x + \tan x| + C_1 \cos x + C_2 \sin x, \end{aligned}$$

where C_1, C_2 are arbitrary constants.